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Stationary definition of persistence for finite-temperature phase ordering

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Abstract. For the two-dimensional kinetic Ising model at finite temperature, the local mean magnetization $M_t = t^{-1}\int_0^t \sigma(t')\,\mathrm{d}t'$, simply related to the fraction of time spent by a given spin in the positive direction, has a limiting distribution, singular at $\pm m_0(T)$, the Onsager spontaneous magnetization. The exponent of this singularity defines the persistence exponent θ. We also study first passage exponents associated to persistent large deviations of M_t, and their temperature dependence.

In this paper we present a new approach to the study of persistence for systems undergoing phase ordering [1, 2] at finite temperature, which we shall illustrate in the case of the two-dimensional Ising model. In this approach persistence appears as a stationary property of the coarsening system, and the role of the Onsager spontaneous magnetization at equilibrium $m_0(T)$ is made apparent, thus revealing new fundamental features of phase ordering. It departs from previous approaches to finite-temperature persistence [3–6], where these features did not appear.

Consider a system of Ising spins $\sigma_i(t) = \pm 1$ located at sites $i = 1, \ldots, N$, starting from a random initial condition, and evolving under the heat bath dynamics at fixed temperature below the critical temperature. At each time-step a spin is picked at random, and updated with the probability

$$P(\sigma_i(t+\mathrm{d}t) = +1) = \frac{1}{2}\left(1 + \tanh\frac{1}{T}\sum_j \sigma_j(t)\right) \tag{1}$$

where the sum runs over the neighbours of site i. Under this dynamics spins thermalize in their local environment. Therefore the system coarsens, i.e. domains of opposite signs grow and, in the scaling regime, the system is statistically self-similar, with only one single characteristic length scale, which is the size of a typical domain [1, 2].

The question of persistence is to determine the fraction of space $R(t)$ which remains in the same phase up to time t [7, 8] (or from time t_1 to time t_2). For the two-dimensional Ising model at zero temperature, two phases coexist, corresponding to all spins equal to $+1$ or all spins equal to -1. Hence $R(t)$ is equivalently defined as the fraction of spins which do not flip up to time t [9], i.e. which were not swept by an interface between domains of all spins $+1$ or all spins -1. Numerical measurements indicate an algebraic decay $R(t) \sim t^{-\theta}$, with

$\theta \approx 0.22$ [9–11]. The same behaviour was observed in an experiment on a liquid crystal system with effective zero temperature two-dimensional Glauber dynamics [12], while a variational estimate of θ at zero temperature leads to a value close to 0.2 [13].

The definition of persistence at finite temperature below T_c is more subtle to implement because one has to make clear what is meant by 'phase'. In essence, in the coarsening process there is phase separation, each phase wanting to develop at the expense of the other. It is therefore intuitive that the system, though perpetually out of equilibrium, tries to reach locally one of the two equilibrium phases, corresponding to $\pm m_0(T)$, where

$$m_0(T) = \left(1 - \left(\sinh \frac{2}{T}\right)^{-4}\right)^{1/8} \tag{2}$$

is the Onsager spontaneous magnetization at equilibrium [14]. Hence in the scaling regime the average magnetization inside a domain measured on a scale of time small compared with the flipping time of the domain, should be close to the equilibrium magnetization at this temperature. Coming back to persistence, the definition of $R(t)$ should reflect, in one way or another, the fact that a given point in space remains in a phase of average magnetization equal to $\pm m_0(T)$ up to time t. This intuitive analysis is confirmed by what follows.

The central point of our approach is to consider the statistics of the local mean magnetization—simply related to the fraction of time spent by a spin in the positive direction—in the limit of large times. This line of thought is used in [15] in the study of domain coarsening for the one-dimensional Ising model at zero temperature, and for the simple diffusion equation evolved from a random initial condition (see also [16]). The idea is that since persistence probes the past history of the system, a natural quantity to consider is $\int_0^t \sigma(t')\,\mathrm{d}t' = T_t^+ - T_t^-$, where $\sigma(t)$ is the spin at site i and T_t^+ (T_t^-) is the length of time spent by the spin pointing upward (downward), with $t = T_t^+ + T_t^-$. The local mean magnetization is defined as

$$M_t = \frac{1}{t}\int_0^t \sigma(t')\,\mathrm{d}t' = 2\frac{T_t^+}{t} - 1. \tag{3}$$

For instance, at zero temperature, the persistence probability $R(t)$ is equal to $\mathcal{P}(M_t = 1)$ since the event $\{\sigma(t') = 1, \forall t' \leqslant t\}$ is identical to the event $\{M_t = 1\}$. Since M_t is a local quantity varying from site to site, one is naturally led to investigate the distribution of M_t,

$$P(t, x) = \mathcal{P}(M_t \geqslant x) \qquad (-1 \leqslant x \leqslant 1). \tag{4}$$

For the one-dimensional Ising model at $T = 0$ it is shown in [15] by analytical arguments and numerical measurements that, when $t \to \infty$, $P(t, x)$ converges to a limit distribution $P_\infty(x)$ with density

$$f_M(x) = -\frac{\mathrm{d}}{\mathrm{d}x} P_\infty(x) \tag{5}$$

singular at $x = \pm 1$, with singularity exponent equal to $\theta - 1$. It is, for example easy to show that, when $t \to \infty$, the limit of $\langle M_t^2 \rangle$ is a constant equal to $\hat{A}(1)$, the Laplace transform of the two-time correlation with respect to the variable $\ln t$, at argument equal to 1 [15]. This result therefore provides a *stationary* definition of persistence at zero temperature. The same holds for the diffusion equation [15, 16].

We now address the same questions at finite temperature.We first report on numerical results. Figure 1 depicts the histogram of the density of M_t at time $t = 1000$ for values of T ranging from 0 to $1.1T_c$. Already for such a short time, and for every temperature $T < T_c$, the density is maximum around $\pm m_0(T)$, the equilibrium magnetization (2). At larger times and for $T > T_c$, the density of M_t becomes peaked around zero, i.e. the mean

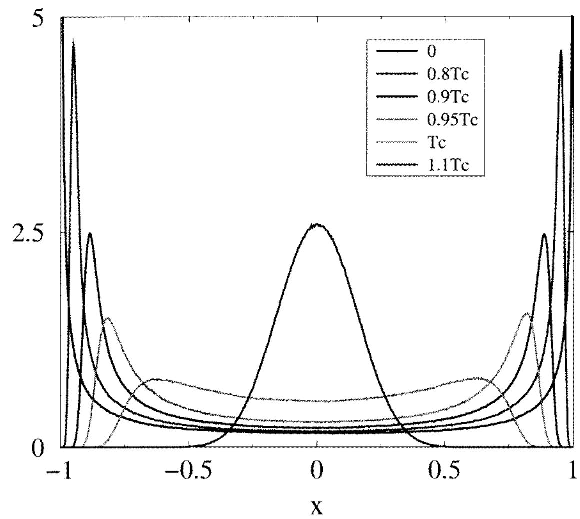

Figure 1. Histogram of the density of M_t at time $t = 1000$, for $T = 0, 0.8T_c$, $0.9T_c$, $0.95T_c$, T_c, $1.1T_c$, from bottom to top (at the central point $x = 0$). (The system size is 3072^2.)

magnetization converges toward the average magnetization per spin $\langle\sigma\rangle = 0$, reflecting the fact that the system reaches equilibrium. At T_c the peaking of the density of M_t is observed to be very slow. Finally at $T < T_c$, $P(t, x)$ converges, when $t \to \infty$, to a limit distribution $P_\infty(x)$ with density $f_M(x)$ given as in (5).

The existence of a limit law at finite temperature $T < T_C$ relies on the same arguments as for the zero-temperature case. For instance, the convergence of $\langle M_t^2\rangle$ to a constant equal to $\hat{A}(1)$ still holds since it only relies on the existence of a scaling regime [15]. The striking fact is that now the density concentrates on $[-m_0(T), m_0(T)]$ with an exponential decay with time of $\mathcal{P}(M_t > m_0(T))$ to 0. Moreover, the limit density is singular at $\pm m_0(T)$, with a singularity exponent $\theta - 1$ which defines the persistence exponent at temperature T.

This leads to the question of the temperature dependence of persistence for $T < T_c$. The simplest assumption is that, during the coarsening process, the timescales between a short-time regime and the scaling regime decouple, yielding the following relation between the moments of the limit distributions at T and at zero temperature,

$$\langle M^{2k}\rangle_T = (m_0(T))^{2k}\langle M^{2k}\rangle_0 \tag{6}$$

implying the identity of the limit distributions, if M_t is rescaled by $m_0(T)$, and as a consequence, the temperature independence of θ. This would be in agreement with the usual view that zero temperature is an attracting fixed point for the dynamics of phase ordering [2, 4]. Equation (6) is hard to check by numerical measurements because the convergence of the data is observed to be slow. The difficulty is illustrated by figure 2 which depicts a plot of $1 - P(t, x)$ against the rescaled variable $x/m_0(T)$, at $T = 0$ and $0.8T_c$, for $t = 20\,000$, and at $T = 0.98T_c$, for $t = 1000, 5000$ and $30\,000$. Though one cannot be conclusive on the sole basis of numerical measurements, data collapse nevertheless seems plausible. Note that the limit distribution $1 - P_\infty(x)$ at $T = 0$ is well approximated by a beta distribution, as was observed for the one-dimensional Ising model [15], or for the diffusion equation [15, 16]. The singularity exponent of the beta distribution is found to be around 0.22. A more precise numerical determination of the exponent θ from the limit distribution of M_t needs further work and will be presented elsewhere.

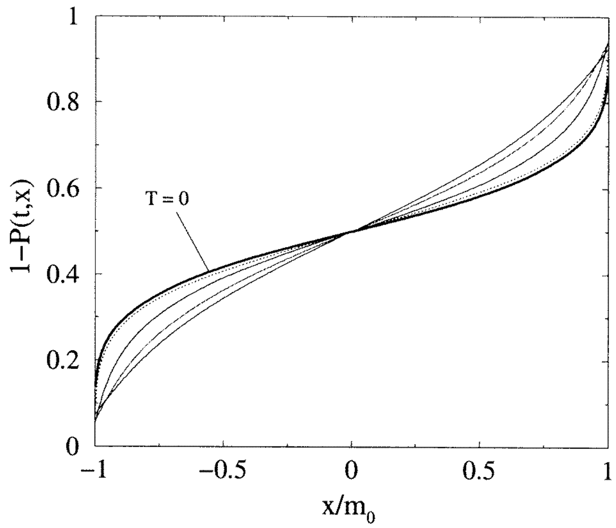

Figure 2. Plot of $1 - P(t, x)$ against the rescaled variable $x/m_0(T)$, at $T = 0$ (full), $T = 0.8T_c$ (dots), for $t = 20\,000$, and at $T = 0.98T_c$, for $t = 1000$, 5000 and 30 000. (The system size is 3072^2.)

Let us summarize at this point. For $T < T_c$, the local mean magnetization M_t has a limit probability density when $t \to \infty$, defined on the interval $[-m_0(T), m_0(T)]$, where $m_0(T)$ is the equilibrium magnetization, and singular at both ends. This provides a *stationary* definition of persistence, which is a natural extension of the zero-temperature case, where the singularity exponent defines the persistence exponent. These are the central results of this work.

Another new aspect of persistence introduced in [15] is concerned with *persistent large deviations*. The probability of persistent large deviations above the level x $(-1 \leqslant x \leqslant 1)$, denoted by $R(t, x)$, is defined as the probability that the mean magnetization was, for all previous times, greater than x [15],

$$R(t, x) = \mathcal{P}(M_{t'} \geqslant x, \forall t' \leqslant t). \tag{7}$$

In other words one is interested in the persistence probability of the stochastic process $\sigma(t, x) = \mathrm{sign}(M_t - x)$ [15]. If one views the stochastic process $\sigma(t)$ as the successive steps of a fictitious random walker, then M_t is the mean speed of the walker between 0 and t, and $R(t, x)$ is the probability that this mean speed remains larger than x between 0 and t. This probability is a natural generalization of the persistence probability $R(t) \equiv R(t, 1)$, which corresponds to the walker always stepping to the right.

For the one-dimensional Ising model at zero temperature, $R(t, x)$ was observed to decay algebraically at large times with an exponent $\theta(x)$ continuously varying with x [15]. For $x = 1$, $\theta(1) = \theta$, the usual persistence exponent. Figure 3 depicts a log–log plot of $R(t, x)$ for the two-dimensional Ising model at zero temperature, x varying from -1 to 1, while figure 4 depicts the corresponding exponents $\theta(x)$, extracted from figure 3. Let us mention that algebraic decay of $R(t, x)$ was also observed for the diffusion equation [15], and that this quantity and the corresponding exponents $\theta(x)$ can be exactly computed for the simple model considered in [17].

We now address the role of temperature for persistent large deviations. As is obvious from the first part of this work, if $x > m_0(T)$, then $R(t, x)$ decays to zero exponentially.

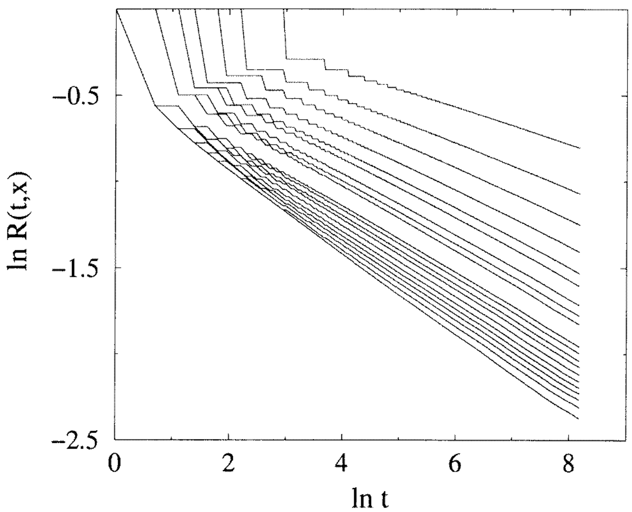

Figure 3. Log–log plot of $R(t, x)$ for the two-dimensional Ising model at zero temperature, x varying from -1 to 1, by steps of 0.1, from top to bottom. (The system size is 1536^2.)

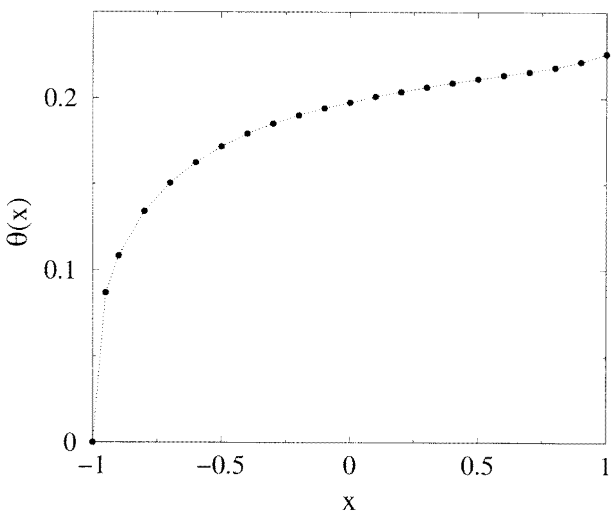

Figure 4. Exponents $\theta(x)$ extracted from figure 3.

On the other hand, for $x < m_0(T)$ one observes algebraic decay of $R(t, x)$, as at zero temperature. Otherwise stated, $x = m_0(T)$ separates two regimes of persistent large deviations, between exponential and algebraic. As a consequence, and by analogy with the zero-temperature case, one could think of extracting the persistence exponent at finite temperature from the decay at large times of $R(t, x)$ when $x \to m_0(T)^-$, which leads to the formal definition

$$R(t) = \lim_{x \to m_0(T)^-} R(t, x) \qquad \text{i.e. } \theta = \lim_{x \to m_0(T)^-} \theta(x). \tag{8}$$

However, this definition is not easy to implement in practice.

We did not investigate the temperature dependence of $\theta(x)$ in all generality. We

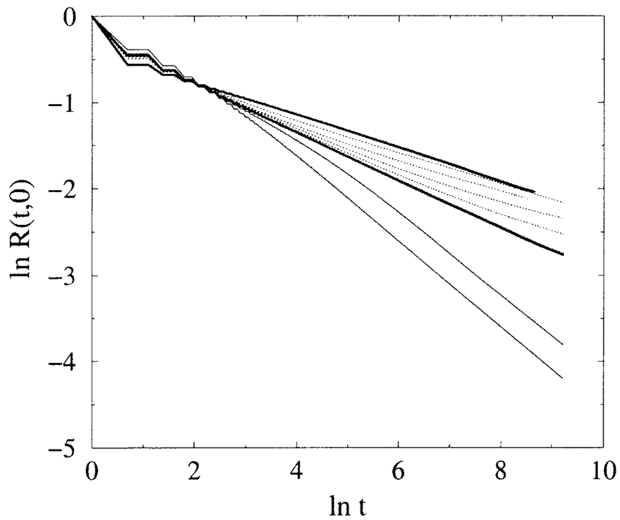

Figure 5. Log–log plot of $R(t, 0)$ for temperatures $T = 0, 0.8T_c, 0.9T_c, 0.95T_c, 0.98T_c, T_c, 1.1T_c, 1.5T_c$, from top to bottom. (The system size is 1536^2.)

restricted our study to the case $x = 0$ which corresponds to

$$R(t,0) = \mathcal{P}\left(\int_0^t \sigma(t')\,\mathrm{d}t' \geqslant 0, \forall t' \leqslant t\right) = \mathcal{P}(T_{t'}^+ \geqslant T_{t'}^-, \forall t' \leqslant t). \tag{9}$$

We find that $R(t, 0) \sim t^{-\theta(0)}$ with

$$\begin{aligned} \theta(0) &= 0.5 && T = \infty \\ \theta(0) &\approx 0.27 && T = T_c \\ \theta(0) &\approx 0.20 && T = 0. \end{aligned} \tag{10}$$

For $T_c < T$, after a crossover, $\theta(0)$ takes the high-temperature value, 0.5, while for $T < T_c$ it takes the low-temperature value, ≈ 0.20. (See figure 5.) The explanation of the value of $\theta(0)$ for $T = \infty$ is simple. Since spins are independent, identifying as above $\sigma(t)$, the spin at site i, to the steps of a fictitious one-dimensional symmetric random walker, $R(t, 0)$ represents the probability that the walker did not cross the origin up to time t, which is, as is well known, decaying as $t^{-1/2}$ [18]. For decreasing temperatures, spins become more correlated, hence the exponent $\theta(0)$ decreases. Note that the first passage exponent $\theta(0)$ is defined for $T \geqslant T_c$, i.e. even in the absence of coarsening.

This work raises a number of questions. For instance, what is the temperature dependence of the two-time correlation in the scaling regime, for $T < T_c$? Is hypothesis (6) valid? What is the behaviour of the distribution of M_t at T_c when $t \to \infty$? At T_c, is $\theta(0)$ a new independent critical exponent, or is it related (equal?) to the persistence exponent θ_c for the global magnetization [19, 4]? Let us mention that, for the three-dimensional Ising model, the quantities studied here have similar behaviour. Finally, in our view, an important point of the analysis presented here is that it may be applied to any coarsening system, since it relies mainly on scaling.

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